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**LETTER TO THE EDITOR**

# A formula for gauge invariant reduction of electromagnetic multipole tensors

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**Abstract**
Based on some previous results, a general formula is given for introducing electromagnetic multipole expansions in terms of symmetric and traceless Cartesian tensors.

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1. Some advantages of the Cartesian forms for multipole moments in the traditional formulation of the electromagnetic theory are well known but the procedure of obtaining multipole tensors corresponding to irreducible representations of the three-dimensional rotation group, especially in the dynamic case, was somehow neglected. However, the method presented in [1] for obtaining the symmetric and traceless part of an $n$th-rank tensor may be successfully used for this aim.

2. Let us consider charge $\rho(\boldsymbol{r}, t)$ and current $\boldsymbol{j}(\boldsymbol{r}, t)$ distributions having supports included in a finite domain $\mathcal{D}$. Choosing the origin $O$ of the Cartesian coordinates in $\mathcal{D}$, the retarded vector and scalar potentials at a point outside $\mathcal{D}, \boldsymbol{r} = x_i \boldsymbol{e}_i$, are given by the multipolar expansions

$$\begin{aligned}
\frac{4\pi}{\mu_0}\boldsymbol{A}(\boldsymbol{r},t) &= \nabla \times \sum_{n\geqslant 1}\frac{(-1)^{n-1}}{n!}\nabla^{n-1}\cdot\left[\frac{1}{r}\mathsf{M}^{(n)}(t_0)\right] + \sum_{n\geqslant 1}\frac{(-1)^{n-1}}{n!}\nabla^{n-1}\cdot\left[\frac{1}{r}\dot{\mathsf{P}}^{(n)}(t_0)\right]\\
&= \boldsymbol{e}_i\varepsilon_{ijk}\partial_j\sum_{n\geqslant 1}\frac{(-1)^{n-1}}{n!}\partial_{i_1}\cdots\partial_{i_{n-1}}\left[\frac{\mathsf{M}_{i_1\cdots i_{n-1},k}(t_0)}{r}\right]\\
&\quad + \boldsymbol{e}_i\sum_{n\geqslant 1}\frac{(-1)^{n-1}}{n!}\partial_{i_1}\cdots\partial_{i_{n-1}}\left[\frac{\dot{\mathsf{P}}_{i_1\cdots i_{n-1}i}(t_0)}{r}\right], \qquad t_0 = t - \frac{r}{c},
\end{aligned} \tag{1}$$

$$4\pi\varepsilon_0\Phi(\boldsymbol{r},t) = \sum_{n\geqslant 0}\frac{(-1)^n}{n!}\nabla^n\cdot\left[\frac{\mathsf{P}^{(n)}(t_0)}{r}\right] = \sum_{n\geqslant 0}\frac{(-1)^n}{n!}\partial_{i_1}\cdots\partial_{i_n}\left[\frac{\mathsf{P}_{i_1\cdots i_n}(t_0)}{r}\right].$$

 

The electric and magnetic moments are defined by

$$\begin{aligned}
\mathsf{P}^{(n)}(t) &= \int_{\mathcal{D}} \boldsymbol{\xi}^n \rho(\boldsymbol{\xi}, t)\,\mathrm{d}^3\xi : \mathsf{P}_{i_1\cdots i_n} = \int_{\mathcal{D}} \xi_{i_1}\cdots\xi_{i_n}\rho(\boldsymbol{\xi}, t)\,\mathrm{d}^3\xi \\
\mathsf{M}^{(n)}(t) &= \frac{n}{n+1}\int_{\mathcal{D}} \boldsymbol{\xi}^n \times \boldsymbol{j}(\boldsymbol{\xi}, t)\,\mathrm{d}^3\xi : \mathsf{M}_{i_1\cdots i_n} = \frac{n}{n+1}\int_{\mathcal{D}} \xi_{i_1}\cdots\xi_{i_{n-1}}(\boldsymbol{\xi}\times\boldsymbol{j})_{i_n}\mathrm{d}^3\xi.
\end{aligned} \tag{2}$$

It is shown in [2] that we can introduce such transformations of the multipole tensors

$$\mathsf{P}^{(n)} \longrightarrow \widetilde{\mathsf{P}}^{(n)}, \qquad \mathsf{M}^{(n)} \longrightarrow \widetilde{\mathsf{M}}^{(n)}, \tag{3}$$

where $\widetilde{\mathsf{P}}^{(n)}$, $\widetilde{\mathsf{M}}^{(n)}$ are fully symmetric and traceless tensors so that, if $\widetilde{\boldsymbol{A}}$ and $\widetilde{\Phi}$ are obtained from equations (1) by the substitutions (3), the correspondence $(\boldsymbol{A}, \Phi) \longrightarrow (\widetilde{\boldsymbol{A}}, \widetilde{\Phi})$ is a gauge transformation.

The present letter is an attempt to systematize some results of the author and co-workers in this field, resumming them in a compact formula.

3. We present the results from [2] by a method initiated in [3][1]. In the transformations (3), the operations of obtaining the symmetric and traceless part of some tensors are implied. Let an $n$th-rank tensor $\mathsf{L}^{(n)}$ of magnetic type, i.e., symmetric in the first $n-1$ indices and verifying the property $\mathsf{L}_{i_1\cdots i_{k-1}ji_{k+1}\cdots i_{n-1}j} = 0, k = 1, \ldots, n-1$. Then, the symmetric part of this tensor is given by

$$\mathsf{L}_{(\mathrm{sym})i_1\cdots i_n} = \frac{1}{n}\left[\mathsf{L}_{i_1\cdots i_n} + \mathsf{L}_{i_ni_2\cdots i_1} + \cdots + \mathsf{L}_{i_1\cdots i_ni_{n-1}}\right] = \mathsf{L}_{i_1\cdots i_n} - \frac{1}{n}\sum_{\lambda=1}^{n-1}\varepsilon_{i_\lambda i_n q}\mathcal{N}^{(\lambda)}_{i_1\cdots i_{n-1}q}[\mathsf{L}^{(n)}]$$

where $\mathcal{N}^{\cdots(\lambda)}_{\cdots}$ is the component with the index $i_\lambda$ suppressed.

The operator $\mathcal{N}$ defines a correspondence between $\mathsf{L}^{(n)}$ and an $(n-1)$th-rank tensor:

$$\mathsf{L}^{(n)} \longrightarrow \mathcal{N}[\mathsf{L}^{(n)}] : [\mathcal{N}[\mathsf{L}^{(n)}]]_{i_1\cdots i_{n-1}} \equiv \mathcal{N}_{i_1\cdots i_{n-1}}[\mathsf{L}^{(n)}] = \varepsilon_{i_{n-1}ps}\mathsf{L}_{i_1\cdots i_{n-2}ps} \tag{4}$$

which is fully symmetric in the first $n-2$ indices and the contractions of the last index with the preceding indices give null results. So, the tensor $\mathcal{N}[\mathsf{L}^{(n)}]$ is of the type $\mathsf{M}^{(n-1)}$. Particularly,

$$\begin{aligned}
\mathcal{N}^{2k}[\mathsf{M}^{(n)}] &= \frac{(-1)^k n}{n+1}\int_{\mathcal{D}} \xi^{2k}\boldsymbol{\xi}^{n-2k} \times \boldsymbol{j}\,\mathrm{d}^3\xi, \\
\mathcal{N}^{2k+1}[\mathsf{M}^{(n)}] &= \frac{(-1)^k n}{n+1}\int_{\mathcal{D}} \xi^{2k}\boldsymbol{\xi}^{n-2k-1} \times (\boldsymbol{\xi}\times\boldsymbol{j})\,\mathrm{d}^3\xi, \qquad k = 0, 1, 2 \ldots.
\end{aligned} \tag{5}$$

Consider a fully symmetric tensor $\mathsf{S}^{(n)}$ and the *detracer* operator $\mathcal{T}$ introduced in [1]. This operator acts on a totally symmetric tensor $\mathsf{S}^{(n)}$ so that $\mathcal{T}[\mathsf{S}^{(n)}]$ is a fully symmetric and traceless tensor of rank $n$. The *detracer theorem* states that [1][2]

$$[\mathcal{T}[\mathsf{S}^{(n)}]]_{i_1\cdots i_n} = \sum_{m=0}^{[n/2]}\frac{(-1)^m(2n-1-2m)!!}{(2n-1)!!}\sum_{D(i)}\delta_{i_1i_2}\cdots\delta_{i_{2m-1}i_{2m}}\mathsf{S}^{(n:m)}_{i_{2m+1}\cdots i_n} \tag{6}$$

where $[n/2]$ denotes the integer part of $[n/2]$, the sum over $D(i)$ is the sum over all permutations of the indices $i_1\cdots i_n$ which give distinct terms and $\mathsf{S}^{(n:m)}_{i_{2m+1}\cdots i_n}$ denotes the components of the $(n-2m)$th-order tensor obtained from $\mathsf{S}^{(n)}$ by contracting $m$ pairs of symbols $i$. Because the number of terms in the sum over $D(i)$ is $n!/2^m m!(n-2m)!$, we have

$$\sum_{D(i)}\delta_{i_1i_2}\cdots\delta_{i_{2m-1}i_{2m}}\mathsf{S}^{(n:m)}_{i_{2m+1}\cdots i_n} = \frac{1}{2^m m!(n-2m)!}\sum_{(\mathcal{P})}\delta_{i_1i_2}\cdots\delta_{i_{2m-1}i_{2m}}\mathsf{S}^{(n:m)}_{i_{2m+1}\cdots i_n}$$

where the last sum is extended to all the permutations of the $i$-indices.

[1] In [3] the demonstrations are given using the charge and current density expansions.

[2] In this equation, the definition of the symmetric and traceless part of the tensor $S^{(n)}$ differs from that used in [1] by a factor $1/(2n-1)!!$.

It is useful to introduce here another operator $\boldsymbol{\Lambda}$ by the equation

$$[\mathcal{T}[\mathbf{S}^{(n)}]]_{i_1\cdots i_n} = \mathsf{S}_{i_1\cdots i_n} - \sum_{D(i)} \delta_{i_1 i_2}[\boldsymbol{\Lambda}[\mathbf{S}^{(n)}]]_{i_3\cdots i_n} \tag{7}$$

where $\boldsymbol{\Lambda}[\mathbf{S}^{(n)}]$ define a fully symmetric tensor of rank $n-2$. From this definition together with the theorem (6), we obtain

$$\Lambda_{i_1\cdots i_{n-2}}[\mathbf{S}^{(n)}] = \sum_{m=0}^{[n/2-1]} \frac{(-1)^m[2n-1-2(m+1)]!!}{(m+1)(2n-1)!!} \sum_{D(i)} \delta_{i_1 i_2}\cdots\delta_{i_{2m-1}i_{2m}} \mathsf{S}^{(n:m+1)}_{i_{2m+1}\cdots i_{n-2}}. \tag{8}$$

In the following, for simplifying the notation, any argument of the operator $\boldsymbol{\Lambda}$ is considered as a symmetrized tensor, i.e., $\Lambda[\mathbf{T}^{(n)}] = \Lambda\big[\mathbf{T}^{(n)}_{\mathrm{sym}}\big]$ for any tensor $\mathbf{T}^{(n)}$. The same observation applies to the operator $\mathcal{T}$: $\mathcal{T}[\mathbf{T}^{(n)}] = \mathcal{T}\big[\mathbf{T}^{(n)}_{\mathrm{sym}}\big]$.

The following four transformation properties of the multipole tensors and potentials may be used for establishing the results from [2].

I. Let the transformation of the $n$th-order magnetic tensor be

$$\mathbf{M}^{(n)} \to \mathbf{M}^{(n)}_{(L)} : \mathsf{M}_{(L)i_1\cdots i_n} = \mathsf{M}_{i_1\cdots i_n} - \frac{1}{n}\sum_{\lambda=1}^{n-1} \varepsilon_{i_\lambda i_n q}\mathcal{N}^{(\lambda)}_{i_1\cdots i_{n-1}q}[\mathbf{L}^{(n)}(t_0)]. \tag{9}$$

Let us substitute in the expansion of the potential $\boldsymbol{A}$ the tensor $\mathbf{M}^{(n)}$ by $\mathbf{M}^{(n)}_{(L)}$ obtaining

$$\begin{aligned}
\boldsymbol{A}\big[\mathbf{M}^{(n)} \to \mathbf{M}^{(n)}_{(L)}\big] &= \boldsymbol{A} - \frac{\mu_0}{4\pi}\frac{(-1)^{n-1}}{n!n}\boldsymbol{e}_i\partial_j\partial_{i_1}\cdots\partial_{i_{n-1}}\left[\frac{1}{r}\sum_{\lambda=1}^{n-1}\varepsilon_{ijk}\varepsilon_{i_\lambda kq}\mathcal{N}^{(\lambda)}_{i_1\cdots i_{n-1}q}[\mathbf{L}^{(n)}(t_0)]\right] \\
&= \boldsymbol{A} + \frac{\mu_0}{4\pi}\frac{(-1)^{n-1}}{n!n}\boldsymbol{e}_i\partial_j\partial_{i_1}\cdots\partial_{i_{n-1}}\frac{1}{r}\big[\big(\delta_{ii_1}\mathcal{N}_{i_2\cdots i_{n-1}j} + \cdots + \delta_{ii_{n-1}}\mathcal{N}_{i_1\cdots i_{n-2}j}\big) \\
&\quad - \big(\delta_{ji_1}\mathcal{N}_{i_2\cdots i_{n-1}i} + \cdots + \delta_{ji_{n-1}}\mathcal{N}_{i_1\cdots i_{n-2}i}\big)\big][\mathbf{L}^{(n)}(t_0)] \\
&= \boldsymbol{A} + \nabla\Psi(\boldsymbol{r},t) + \frac{\mu_0}{4\pi}\frac{(-1)^n(n-1)}{n!c^2n}\nabla^{n-2}\cdot\left[\frac{1}{r}\ddot{\boldsymbol{\mathcal{N}}}[\mathbf{L}^{(n)}(t_0)]\right].
\end{aligned} \tag{10}$$

Here, the relations $\varepsilon_{ijk}\varepsilon_{i_\lambda kq} = \delta_{iq}\delta_{ji_\lambda} - \delta_{ii_\lambda}\delta_{jq}$ and $[\Delta - (1/c^2)\partial^2/\partial t^2][f(t_0)/r] = 0, r \neq 0$ are considered. The function $\Psi$ is a solution of the homogeneous wave equation for $r \neq 0$ and the corresponding expression is irrelevant. Make the transformation

$$\mathbf{P}^{(n-1)} \to \mathbf{P}'^{(n-1)} = \mathbf{P}^{(n-1)} + a_1(n)\dot{\boldsymbol{\mathcal{N}}}[\mathbf{L}^{(n)}], \qquad a_1(n) = -\frac{n-1}{c^2n^2}. \tag{11}$$

Introducing the transformed potentials produced by the substitution $\mathbf{P}^{(n-1)} \to \mathbf{P}'^{(n-1)}$, we obtain

$$\boldsymbol{A}\big[\mathbf{M}^{(n)} \to \mathbf{M}^{(n)}_{(L)}, \mathbf{P}^{(n-1)} \to \mathbf{P}'^{(n-1)}\big] = \boldsymbol{A} + \nabla\Psi, \qquad \Phi[\mathbf{P}^{(n-1)} \to \mathbf{P}'^{(n-1)}] = \Phi - \frac{\partial\Psi}{\partial t}.$$

So, the transformation (9) produces changes in the potentials which, up to a gauge transformation, are compensated by the transformation (11).

II. Let the transformation of the $n$th-order electric tensor be

$$\mathbf{P}^{(n)} \to \mathbf{P}^{(n)}_{(L)} : \mathsf{P}_{(L)i_1\cdots i_n} = \mathsf{P}_{i_1\cdots i_n} - \frac{1}{n}\sum_{\lambda=1}^{n-1}\varepsilon_{i_\lambda i_n q}\mathcal{N}^{(\lambda)}_{i_1\cdots i_{n-1}q}[\mathbf{L}^{(n)}(t_0)]. \tag{12}$$

We obtain

$$\boldsymbol{A}\big[\mathbf{P}^{(n)} \to \mathbf{P}^{(n)}_{(L)}\big] = \boldsymbol{A} + \frac{\mu_0}{4\pi}\frac{(-1)^{n-1}(n-1)}{n!n}\nabla\times\left\{\nabla^{n-2}\cdot\left[\frac{1}{r}\dot{\boldsymbol{\mathcal{N}}}\big[\mathbf{L}^{(n)}\big]\right]\right\},$$

$$\Phi\big[\mathbf{P}^{(n)} \to \mathbf{P}^{(n)}_{(L)}\big] = \Phi.$$

The change of the vector potential is compensated by the transformation

$$\mathbf{M}^{(n-1)} \longrightarrow \mathbf{M}^{(n-1)} + a_2(n)\dot{\mathcal{N}}[\mathbf{L}^{(n)}], \qquad a_2(n) = \frac{n-1}{n^2} = -c^2 a_1(n). \tag{13}$$

III. Let the transformation of the magnetic vector of rank $n$ be

$$\mathbf{M}^{(n)} \longrightarrow \mathbf{M}^{(n)}_{(S)} : \mathbf{M}_{(S)i_1\cdots i_n} = \mathbf{M}_{i_1\cdots i_n} - \sum_{D(i)} \delta_{i_1 i_2} \Lambda_{i_3\cdots i_n}[\mathbf{S}^{(n)}(t_0)] \tag{14}$$

where $\mathbf{S}^{(n)}$ is a fully symmetric tensor.

The change in the vectorial potential produced by this transformation is

$$\boldsymbol{A}\big[\mathbf{M}^{(n)} \to \mathbf{M}^{(n)}_{(S)}\big] = \boldsymbol{A} - \frac{\mu_0}{4\pi}\frac{(-1)^{n-1}(n-2)(n-1)}{2c^2 n!}\nabla \times [\nabla^{n-3} \cdot \ddot{\Lambda}[\mathbf{S}^{(n)}]].$$

This alteration of the vectorial potential is eliminated by the transformation

$$\mathbf{M}^{(n-2)} \longrightarrow \mathbf{M}^{(n-2)} + b(n)\ddot{\Lambda}[\mathbf{S}^{(n)}], \qquad b(n) = \frac{n-2}{2c^2 n}. \tag{15}$$

IV. The transformation

$$\mathbf{P}^{(n)} \longrightarrow \mathbf{P}_{(S)} : \mathbf{P}_{(S)i_1\cdots i_n} = \mathbf{P}_{i_1\cdots i_n} - \sum_{D(i)} \delta_{i_1 i_2} \Lambda_{i_3\cdots i_n}[\mathbf{S}^{(n)}] \tag{16}$$

produces the following changes of the potentials:

$$\begin{aligned}\boldsymbol{A}\big[\mathbf{P}^{(n)} \to \mathbf{P}^{(n)}_{(S)}\big] &= \boldsymbol{A} - \frac{\mu_0}{4\pi}\frac{(-1)^{n-1}}{n!} e_i \partial_{i_1}\cdots\partial_{i_{n-1}}\left[\frac{1}{r}\sum_{D(i)} \delta_{i_1 i_2}\dot{\Lambda}_{i_3\cdots i_n}[\mathbf{S}^{(n)}]\right] \\ &= \boldsymbol{A} + \nabla\Psi' - \frac{\mu_0}{4\pi}\frac{(-1)^{n-1}(n-2)(n-1)}{2n!c^2}\nabla^{n-3}\cdot\left[\frac{1}{r}\dddot{\Lambda}[\mathbf{S}^{(n)}]\right]\end{aligned}$$

with $\Psi'$, as $\Psi$, satisfying the homogeneous wave equation and

$$\Phi\big[\mathbf{P}^{(n)} \to \mathbf{P}^{(n)}_{(S)}\big] = \Phi + \frac{\mu_0}{4\pi}\frac{(-1)^{n-1}n(n-1)}{2n!}\nabla^{n-2}\cdot\left[\frac{1}{r}\ddot{\Lambda}[\mathbf{S}^{(n)}]\right].$$

Take the transformation

$$\mathbf{P}^{(n-2)} \longrightarrow \mathbf{P}''^{(n-2)} = \mathbf{P}^{(n-2)} + b(n)\ddot{\Lambda}[\mathbf{S}(t_0)] \tag{17}$$

with $b(n)$ given by equation (15). The effect of the transformation (17) on the potential $\boldsymbol{A}$ is the compensation of the extra-gauge term. So $\boldsymbol{A}\big[\mathbf{P}^{(n)} \to \mathbf{P}^{(n)}_{(S)}, \mathbf{P}^{(n-2)} \to \mathbf{P}''^{(n-2)}\big] = \boldsymbol{A} + \nabla\Psi'$ but it is easy to see that the modification of the scalar potential $\Phi$ produced by the transformation (17) together with the modification due to the transformation (16) gives $\Phi\big[\mathbf{P}^{(n)} \to \mathbf{P}^{(n)}[\mathbf{S}^{(n)}], \mathbf{P}^{(n-2)} \to \mathbf{P}''^{(n-2)}\big] = \Phi - \partial\Psi'/\partial t$ the total effect of the transformations (16) and (17) being a gauge transformation of the potentials.

4. Let the gauge-invariant process of reducing the multipole tensors begin for the electric tensors from the order $n = \varepsilon$ and for the magnetic ones from $n = \mu$. Generally, we may choose $\varepsilon > \mu$ as seen, for example, from the calculation of the total power radiated by a confined system of charges and currents [3]. The following formulae are results of the rules represented by equations (11), (13), (15) and (17):

$$\begin{aligned}\widetilde{\mathbf{P}}^{(n)} = \mathcal{P}^{(n)} + \mathcal{T}\Bigg\{&\sum_{k=1}^{[(\varepsilon-n)/2]} A_k^{(n)}\frac{\mathrm{d}^{2k}}{\mathrm{d}t^{2k}}\Lambda^k[\mathbf{P}^{(n+2k)}] \\ &+ \sum_{k=0}^{[(\mu-n-1)/2]}\frac{\mathrm{d}^{2k+1}}{\mathrm{d}t^{2k+1}}\sum_{l=0}^{k} B_{kl}^{(n)}\Lambda^l\mathcal{N}^{2k-2l+1}[\mathbf{M}^{(n+1+2k)}]\Bigg\},\end{aligned} \tag{18}$$

$$\widetilde{\mathbf{M}}^{(n)} = \mathcal{M}^{(n)} + \mathcal{T}\left\{\sum_{k=1}^{[(\mu-n)/2]} \frac{\mathrm{d}^{2k}}{\mathrm{d}t^{2k}} \sum_{l=0}^{k} C_{kl}^{(n)} \boldsymbol{\Lambda}^{l} \boldsymbol{\mathcal{N}}^{2k-2l}[\mathbf{M}^{(n+2k)}]\right\} \tag{19}$$

where

$$\begin{aligned} A_k^{(n)} &= \prod_{l=1}^{k} b(n+2l), \\ B_{kl}^{(n)} &= \prod_{q=1}^{l} b(n+2q) \prod_{h=0}^{k-l} a_1(n+1+2k-2h) \prod_{s=0}^{k-l-1} a_2(n+2k-2s) \end{aligned} \tag{20}$$

and

$$C_{kl}^{(n)} = \prod_{q=1}^{l} b(n+2q) \prod_{h=0}^{k-l-1} a_1(n+2k-2h) \prod_{s=0}^{k-l-1} a_2(n-1+2k-2s). \tag{21}$$

By $\mathcal{P}^{(n)}$ and $\mathcal{M}^{(n)}$ we understand the 'static' expressions of the reduced multipole tensors:

$$\mathcal{P}^{(n)}(t) = \mathcal{T}[\mathbf{P}^{(n)}] = \frac{(-1)^n}{(2n-1)!!} \int_{\mathcal{D}} \rho(\boldsymbol{r}, t) r^{2n+1} \boldsymbol{\nabla}^n \frac{1}{r} \mathrm{d}^3 x,$$

$$\mathcal{M}^{(n)}(t) = \mathcal{T}[\mathbf{M}^{(n)}] = \frac{(-1)^n}{(n+1)(2n-1)!!} \sum\nolimits_{\lambda=1}^{n} \int_{\mathcal{D}} r^{2n+1} [\boldsymbol{j}(\boldsymbol{r}, t) \times \boldsymbol{\nabla}]_{i_\lambda} \partial_{i_1 \cdots i_n}^{(\lambda)} \frac{1}{r} \, \mathrm{d}^3 x.$$

In these formulae one considers $\prod_{k=l}^{L} F_k = 1$ if $L < l$. For justifying the formulae (18), (19), we may consider separately the reduction of the tensors $\mathbf{P}^{(n)}$ beginning from $n = \varepsilon$. Using equation (17), we obtain

$$\mathbf{P}^{(\varepsilon-2p)} \to \widetilde{\mathbf{P}}'^{(\varepsilon-2p)} = \mathcal{P}^{(\varepsilon-2p)} + \mathcal{T}\left[\sum_{k=1}^{p} A_k^{(\varepsilon-2p)} \frac{\mathrm{d}^{2k}}{\mathrm{d}t^{2k}} \boldsymbol{\Lambda}^k [\mathbf{P}^{(\varepsilon-2p+2k)}]\right],$$
$$p = 0, 1, 2, \ldots. \tag{22}$$

Repeating the operations but beginning from $n = \varepsilon - 1$, we obtain the full set of reduced electric tensors for $\mu = 0$. The process of reduction is a little more complicated for the magnetic tensors because of the operation of symmetrization. Beginning with $n = \mu$ and applying equations (11), (13), we obtain the following results for the symmetrized tensors:

$$\mathbf{M}^{(\mu-2p)} \to \mathbf{M}_{\mathrm{sym}}^{(\mu-2p)} + \left\{\sum_{k=1}^{p} C_{k0}^{(\mu-2p)} \frac{\mathrm{d}^{2k}}{\mathrm{d}t^{2k}} \boldsymbol{\mathcal{N}}^{2k} [\mathbf{M}^{(\mu-2p+2k)}]\right\}_{\mathrm{sym}}, \tag{23}$$

$$\widetilde{\mathbf{P}}'^{(\mu-2p-1)} \to \widetilde{\mathbf{P}}'^{(\mu-2p-1)} + \left\{\sum_{k=0}^{p} B_{k0}^{(\mu-2p-1)} \frac{\mathrm{d}^{2k+1}}{\mathrm{d}t^{2k+1}} \boldsymbol{\mathcal{N}}^{2k+1} [\mathbf{M}^{(\mu-2p+2k)}]\right\}_{\mathrm{sym}}. \tag{24}$$

By applying equations (15) and (17) to these results, we obtain equations (18) and (19).

Equations (18) and (19) show that it is possible to give compact formulae for the electromagnetic multipolar expansions using the general tensorial formalism and all is reduced to simple algebraic calculations which may be performed also by automatic numerical or symbolic computation [4].

The gauge-invariant reduction procedure described by the points 1–3 and formulae (18) and (19) present the advantage of simplicity and it is done in a systematic way which will be pointed out in the following.

(a) By counting the contributions of the linear dimensions of the domain $\mathcal{D}$, the tensors $\mathsf{M}^{(n)}$ and $\dot{\mathsf{P}}^{(n+1)}$ are comparable [3]. Consequently, in equations (18) and (19) the case $\varepsilon = \mu + 1$ is to be considered. In this case, there is a new way to group terms in equation (18),

$$\begin{aligned} \widetilde{\mathsf{P}}^{(n)} &= \mathcal{P}^{(n)} + \sum_{k=1}^{[(\varepsilon-n)/2]} \frac{(-1)^k}{c^{2k}} \frac{\mathrm{d}^{2k-1}}{\mathrm{d}t^{2k-1}} \mathsf{T}_{(k)}^{(n)}, \\ \mathsf{T}_{(k)}^{(n)} &= (-1)^k c^{2k} \left[ A_k^{(n)} \Lambda^k [\dot{\mathsf{P}}^{(n+2k)}] + \sum_{l=0}^{k-1} B_{k-1,l}^{(k)} \Lambda^l \mathcal{N}^{2k-2l-1} [\mathsf{M}^{(n+2k-1)}] \right], \end{aligned} \tag{25}$$

where besides the usual electric and magnetic multipole moments one displays a third multipole family, the toroid moments and, generally, mean-square radii of various orders.

These results may be compared with those obtained using a different formalism in [5, 6]. As a simple example, $\mathsf{T}_{(1)}^{(1)}$ and $\mathsf{T}_{(2)}^{(1)}$ may be compared with, respectively, the toroid dipole moment $\boldsymbol{t}$ and the first mean-square radius $\vec{\bar{R}}^2$ of the toroid dipole distribution from [6].

As is known, this third family of multipoles is related to problems of violations of spacetime symmetries in elementary particle, atomic, nuclear and molecular physics. Theoretical and experimental results related to the content of equations (19) and (25) are presented in a large number of publications, most of them being cited in the above-mentioned reviews.

(b) Using the reduced tensors $\widetilde{\mathsf{P}}^{(n)}$ and $\widetilde{\mathsf{M}}^{(n)}$ one obtains, by simple algebraic calculations implying only elementary tensorial manipulations and some combinatorics, results for various physical quantities such as radiation power, angular momentum loss, recoil force or interaction energy. One can see, for example, in the case of the radiation intensity, the total power radiated is very simply expressed in terms of these tensors, [7]:

$$\begin{aligned} \mathcal{J}_{\mu\varepsilon} = \frac{1}{4\pi\varepsilon_0 c^3} \Bigg[ &\sum_{n=1}^{\mu} \frac{n+1}{nn!(2n+1)!!c^{2n}} \left(\widetilde{\mathsf{M}}_{,n+1}^{(n)} \big|\big| \widetilde{\mathsf{M}}_{,n+1}^{(n)}\right) \\ &+ \sum_{n=1}^{\varepsilon} \frac{n+1}{nn!(2n+1)!!c^{2n-2}} \left(\widetilde{\mathsf{P}}_{,n+1}^{(n)} \big|\big| \widetilde{\mathsf{P}}_{,n+1}^{(n)}\right) \Bigg] \end{aligned}$$

where $f_{,k}$ denotes the time derivative of the order $k$ of $f$. A similar simple structure characterizes the expressions of the angular momentum loss and recoil forces. The corresponding results will be presented in a future publication. Some issues related to such results are analysed in [3]. For example, by treating an elementary system manifesting itself as a toroid electric dipole, it suffices to calculate separately the corresponding contribution to a physical quantity. On the other hand, if the contribution for a composite system—an atom, a nucleus, etc—is considered, then all the multipolar terms contributing with the same order of magnitude must be calculated together with toroid dipole moment. So, for example, in the treatment of the 'magnetic dipole and electric quadrupole' radiation in the well-known textbooks by Landau and Lifschitz [8] and Jackson [9] the corresponding results are correct for the problem stated there and correspond to the first from the above circumstances, contrary to the criticism formulated in [6]. It seems that in the literature the problem of consistently imposing the criteria according to which different terms of the multipole expansion must be considered in a given approximation is treated in different ways producing contradictory results. So, in [10] a consistent method of approximation is used, but only a part of the toroid dipole is obtained, whereas in [5, 6] the exact contribution of this toroid dipole is obtained but either terms corresponding to the same approximation are neglected, or an incomplete number of terms corresponding to a higher approximation are present.

The main goal of this letter is not to obtain new physical results but to offer a method to treat some complicated physical problems in a systematic, self-consistent way, making use of simple mathematical methods.

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